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# Integrability in theories with local $U(1)$ gauge symmetry

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## Abstract

Using a recently developed method, based on a generalization of the zero curvature representation of Zakharov and Shabat, we study the integrability structure in the Abelian Higgs model. It is shown that the model contains integrable sectors, where integrability is understood as the existence of infinitely many conserved currents. In particular, a gauge invariant description of the weak and strong integrable sectors is provided. The pertinent integrability conditions are given by a  $U(1)$  generalization of the standard strong and weak constraints for models with a two-dimensional target space. The Bogomolny sector is discussed, as well, and we find that each Bogomolny configuration supports infinitely many conserved currents. Finally, other models with  $U(1)$  gauge symmetry are investigated.

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## 1. Introduction

One of the most important features of realistic field theories, both in elementary particle theory and in applications to condensed matter systems, is their nonlinearity. Unfortunately, nonlinear field theories are notoriously difficult to analyse. In low dimensional base spaces, the concept of integrability, i.e. the existence of infinitely many conserved quantities, has proven very useful for the investigation of nonlinear field theories. In higher dimensions, much less is known about nonlinear theories, and a general concept of integrability has not yet been developed there. However, a generalization of the zero-curvature condition of Zakharov and Shabat has been proposed in [1], and has been used to construct nonlinear field theories which have either infinitely many conservation laws in the full theory or which contain integrable subsectors defined by some additional constraint equations on the fields. Applications of this construction have shown that usually these conserved currents in the models or their subsectors are Noether

currents and generalizations thereof, i.e. they are related to the geometry and symmetries of the target space (see, e.g., [2]). So a direct, geometric approach has been successfully undertaken to find these conservation laws, e.g., for models with two- and three-dimensional target space (like, e.g., the Faddeev–Niemi and Skyrme models) [3–7]. Generalization to a higher dimensional target space ( $CP^n$  model) has been established, as well [8, 9]. So far this method has only been used in the investigation of field theories without gauge symmetries. The main aim of the present work is to adopt this approach to some nonlinear field theories with an Abelian gauge symmetry. In particular, we will perform calculations for the Abelian Higgs model. It is worth underlining that our analysis is valid for an arbitrary,  $(d + 1)$ -dimensional spacetime.

Obviously, the main problem in the application of the zero curvature framework to theories with a local gauge symmetry is the gauge invariance of the integrable subsectors. Fortunately, we find that one can define such sectors in a completely gauge-independent manner.

The paper is organized as follows. In section 2, we briefly recall the integrability of a class of non-gauge theories for later convenience. Then we construct the integrability conditions and the corresponding conserved currents for the Abelian Higgs model. Further, we discuss the relation of the integrable sectors to the Bogomolny sector of the model. In section 3, we discuss some generalizations. Section 4 contains our conclusions.

## 2. Integrability in the Abelian Higgs model

The Abelian Higgs model is given by the following Lagrangian density:

$$\mathcal{L} = -F_{\mu\nu}^2 + \frac{1}{2}(D_\mu u)(D^\mu u)^* - V, \quad (1)$$

where the covariant derivative is  $D_\mu u = u_\mu - ieA_\mu u$  and  $u_\mu \equiv \partial_\mu u$ . Moreover,  $F_{\mu\nu}$  is the standard antisymmetric field tensor for the Abelian, real gauge field  $A_\mu$ . The Higgs potential reads

$$V = \frac{\lambda}{4} \left( \frac{m^2}{\lambda} - uu^* \right)^2. \quad (2)$$

Here,  $e$  and  $\lambda$  are gauge and scalar coupling constants, respectively, whereas  $m$  fixes the vacuum value of the Higgs field.

The canonical momenta are

$$\pi_\mu \equiv \frac{\partial L}{\partial u_\mu} = \frac{1}{2}(D_\mu u)^*, \quad \pi_\mu^* \equiv \frac{\partial L}{\partial u_\mu^*} = \frac{1}{2}D_\mu u \quad (3)$$

and

$$\pi_{\mu\nu} \equiv \frac{\partial L}{\partial(\partial^\mu A^\nu)} = -4F_{\mu\nu}. \quad (4)$$

The pertinent equations of motion read

$$\partial_\mu \pi^\mu = L_u = -ieA_\mu \pi^\mu - V'u^*, \quad \partial_\mu \pi^{*\mu} = L_{u^*} = ieA_\mu \pi^{*\mu} - V'u, \quad (5)$$

$$\partial_\mu F^{\mu\nu} = \frac{ie}{8}[(D^\nu u)^*u - (D^\nu u)u^*], \quad (6)$$

where the prime denotes differentiation with respect to  $uu^*$ .

### 2.1. The non-gauge case

Before defining the integrable sectors and the conserved currents in the Abelian Higgs model, we find it convenient to briefly recall the analogous situation for some non-gauge models. Integrability for a large class of theories with a two-dimensional target space given by a complex field  $u$  has been discussed in [5, 6]. Here we just want to quote results for a small subclass of models, which are useful for our purposes. For the class of  $\mathbb{CP}^1$ -type models

$$\mathcal{L}_{\mathbb{CP}^1} = f(uu^*)u^\mu u_\mu^* \quad (7)$$

the currents

$$j_\mu = \frac{i}{f}(G_u \pi_\mu^* - G_{u^*} \pi_\mu) \quad (8)$$

are conserved provided that the additional constraint or integrability condition (the complex eikonal equation)

$$u^\mu u_\mu = 0 \quad (9)$$

holds. Here,  $\pi_\mu$  is the canonical 4-momentum of the  $\mathbb{CP}^1$ -type model (7) and  $G \equiv G(u, u^*)$  is an arbitrary real function of the field  $u$  and its complex conjugate; further,  $G_u \equiv \partial_u G$ . We call this condition the strong integrability condition and the corresponding sector of fields the strong integrable sector.

For the class of  $\mathbb{CP}^1$ -type models with an additional potential

$$\mathcal{L}_{\mathbb{CP}^1, V} = f(uu^*)u^\mu u_\mu^* - V(uu^*), \quad (10)$$

the currents

$$j_\mu = i(G_u \pi_\mu^* - G_{u^*} \pi_\mu) = iG'(uu^*)(u^* \pi_\mu^* - u \pi_\mu) \quad (11)$$

are conserved provided that the additional constraint or integrability condition (the weak integrability condition)

$$u^{*2} u^\mu u_\mu - u^2 u^{*\mu} u_\mu^* = 0 \quad (12)$$

holds. Here it is assumed that  $G \equiv G(uu^*)$  is a function of the modulus squared  $uu^*$  only. Observe that the factor  $f^{-1}$ , which is present in the definition of the conserved currents of the strong sector, is unnecessary for the currents of the weak sector, because it can be compensated by a redefinition of the arbitrary  $G(uu^*)$ .

How can we expect that these results carry over to the case of a gauge theory, specifically to the Abelian Higgs model? Gauge invariant generalizations of the weak and strong integrability conditions are naturally given by

$$u^2 (D_\mu u)^{*2} - u^{*2} (D_\mu u)^2 = 0 \quad (13)$$

(the weak gauge invariant condition), and by

$$(D_\mu u)^2 = 0 \quad (14)$$

(the strong gauge invariant condition), as we shall obtain in the following. We will find, however, that there exists a subtle difference between the weak and strong sectors concerning the currents. The weak currents (11), when defined for the Abelian Higgs model, are gauge invariant (because of  $G = G(uu^*)$ ) and have gauge invariant conservation equations. Their conservation leads therefore directly to the weak condition (13), as we shall see. On the other hand, the currents (8) for the Abelian Higgs model are not gauge invariant and do not give rise to gauge invariant conservation equations (because of  $G = G(u, u^*)$ ). Therefore, the strong condition (14) has to be interpreted differently. We will find that it means the *existence* of a

gauge such that the strong currents (8) are conserved in this gauge. It is this latter condition which is gauge invariant.

We have indicated, right now, the existence of conserved currents in subsectors of the Abelian Higgs model, which are in close analogy with the case of non-gauge theories. One might ask whether there exist further conserved currents in some integrable sectors, which have no analogy in non-gauge theories. Interestingly, the answer to this question is yes, as we shall see shortly.

2.2. *The weak integrable gauge sector*

Taking into account the above discussion, we assume the following family of currents for the Abelian Higgs model:

$$j_\mu = i(G_u \pi_\mu^* - G_{u^*} \pi_\mu), \tag{15}$$

where  $G$  is an arbitrary function of the modulus  $|u|$  squared, that is,  $G = G(uu^*)$ . These currents are conserved if

$$0 = -i\partial_\mu j^\mu = G''(uu_\mu^* + u^*u_\mu)(u^* \pi^{*\mu} - u\pi^\mu) + G'(u_\mu^* \pi^{*\mu} - u_\mu \pi^\mu + u^* \partial^\mu \pi_\mu^* - u \partial^\mu \pi_\mu). \tag{16}$$

After some calculations, one finds that the condition providing conservation of the currents takes the form

$$0 = u^{*2}u_\mu^2 - u^2u_\mu^{*2} - 2ieuu^*[uu_\mu^* + u^*u_\mu]A^\mu. \tag{17}$$

It is easy to note that this equation is the natural generalization of the well-known weak integrable condition for models with a two-dimensional target space [6]. Therefore we call this sector of the Abelian Higgs model, defined by the additional constraint (17), the weak one. The weak integrability condition reveals its geometrical meaning if we reexpress it in terms of the polar decomposition of the Higgs field:

$$u = e^{\Sigma+i\Lambda}, \tag{18}$$

where  $\Sigma, \Lambda$  are real fields. Then we get

$$\Sigma_\mu(\Lambda^\mu - eA^\mu) = 0. \tag{19}$$

Thus, the weak integrable sector is defined by field configurations such that the gradient of the modulus is perpendicular to a covariant derivative of the phase.

As the full theory possesses the  $U(1)$  gauge symmetry, we should also have a gauge invariant description for the integrable sector. One can check that our constraint as well as the currents is invariant under the gauge transformation

$$\Sigma \rightarrow \Sigma, \quad \Lambda \rightarrow \Lambda + ef, \quad A_\mu \rightarrow A_\mu + \partial_\mu f. \tag{20}$$

In fact, we can rewrite expression (17) in an elegant, manifestly gauge-independent way as  $u^2(D_\mu u)^{*2} - u^{*2}(D_\mu u)^2 = 0$ , as announced by equation (13). As said, this is the  $U(1)$  gauge invariant generalization of the standard weak constraint.

Of course, one should ask whether such a submodel has any solution at all. The answer is yes. For example, let us consider the standard static, axially symmetric vortex ansatz in (2+1) dimensions [10],

$$u(\rho, \phi) = e^{in\phi} v(\rho), \quad A_0 = 0, \quad A_1 = -\frac{y}{\rho} A(\rho), \quad A_2 = \frac{x}{\rho} A(\rho), \tag{21}$$

where  $\rho = \sqrt{x^2 + y^2}$  and the functions  $v(\rho), A(\rho)$  still have to be determined from the pertinent ordinary second-order differential equations (a trivial immersion into arbitrary  $d > 2$

space dimensions shows that the pertinent weak sector in higher dimensions is non-empty as well). Such solutions of the Abelian Higgs model with a non-critical coupling are known numerically or in an approximated form [11, 12]. As we see, for this ansatz

$$\Lambda = n\phi, \quad \Sigma = \ln v(\rho). \quad (22)$$

Obviously, these fields obey the weak integrability condition. This fact resembles the results obtained for the  $CP^n$  model, where non-BPS solutions belong to one of the weak sectors [9].

It should be underlined that the weak integrable condition gives the conserved currents for all values of the scalar coupling constant  $\lambda$ .

Finally, let us point out that the currents are conserved *half off-shell*. Indeed, the conservation law is fulfilled if the equations of motion for the Higgs field are imposed. On the other hand, it is not necessary to assume that the gauge field satisfies (6).

### 2.3. New currents in the weak integrable sector

Rather surprisingly, we are able to construct another family of conserved currents in the weak sector. Namely,

$$j_\mu = F_{\mu\nu}(H_u\pi^{*\nu} + H_{u^*}\pi^\nu), \quad (23)$$

where the arbitrary function  $H$  depends on the modulus  $|u|$  squared, i.e.  $H = H(uu^*)$ .

Then,

$$\begin{aligned} \partial_\mu j^\mu &= \frac{H'}{2} \partial_\mu F^{\mu\nu} (u^*u^\nu + uu^{*\nu}) + \frac{1}{2} F_{\mu\nu} [H'(u^{*\mu}u^\nu + u^*u^{\mu\nu} + u^\mu u^{*\nu} + uu^{*\mu\nu}) \\ &\quad + H''(u^\mu u^* + uu^{*\mu})(u^*u^\nu + uu^{*\nu})]. \end{aligned} \quad (24)$$

We easily get

$$\partial_\mu j^\mu = \frac{ieH'}{16} [(D^\nu u)^*u - (D^\nu u)u^*][u^*u^\nu + uu^{*\nu}], \quad (25)$$

or after the polar decomposition

$$\partial_\mu j^\mu = \frac{H'(uu^*)^2}{4} (\Lambda_\nu - eA_\nu)\Sigma^\nu. \quad (26)$$

Due to the fact that the fields belong to the weak integrable sector, this expression vanishes.

Similar as before, this new class of conserved currents is gauge independent and exists for any scalar coupling constant. Moreover, they are also conserved *half off-shell*. However, inversely to the previously discussed case, one must impose the equation of motion for the gauge field while the complex field does not have to obey the field equations.

### 2.4. The strong integrable sector

In order to find more conserved currents, we now set  $\lambda = 0$ , i.e. we neglect the potential term in the Lagrangian, analogous to the non-gauge case discussed in section 2.1.

The currents are given by the same expression as in the weak case,

$$j_\mu = i(G_u\pi_\mu^* - G_{u^*}\pi_\mu), \quad (27)$$

but now the function  $G$  depends on  $u$  and  $u^*$  in a completely arbitrary manner,  $G = G(u, u^*)$ .

Then,

$$-i\partial_\mu j^\mu = G_{uu}u_\mu\pi^{*\mu} + G_{uu^*}u_\mu^*\pi^{*\mu} - G_{u^*u}u_\mu\pi^\mu - G_{u^*u^*}u_\mu^*\pi^\mu + G_u\partial_\mu\pi^{*\mu} - G_{u^*}\partial_\mu\pi^\mu \quad (28)$$

vanishes if

$$u_\mu^* \pi^{*\mu} - u_\mu \pi^\mu = 0, \quad (29)$$

$$u_\mu \pi^{*\mu} = 0, \quad u_\mu^* \pi^\mu = 0, \quad (30)$$

$$A_\mu \pi^{*\mu} = 0, \quad A_\mu \pi^\mu = 0 \quad (31)$$

or, if expressed using the polar decomposition,

$$\Sigma_\mu A^\mu = 0, \quad (32)$$

$$\Sigma_\mu \Lambda^\mu = 0, \quad (33)$$

$$\Sigma_\mu^2 - \Lambda_\mu^2 + e \Lambda_\mu A^\mu = 0, \quad (34)$$

$$A_\mu (\Lambda^\mu - e A^\mu) = 0. \quad (35)$$

We see immediately that these so-called strong integrability conditions are gauge dependent.

To make sense of the strong integrable sector, we have to understand integrability in gauge models as the existence of a gauge in which there are infinitely many conserved currents, that is, the existence of a gauge in which conditions (32)–(35) are fulfilled. Then we may ask another question, namely: what should be the conditions for these fields which would provide the existence of such a gauge?

So, let us take arbitrary fields  $\Sigma', \Lambda', A'_\mu$ . In the most general case, the expressions for the strong constraints (32)–(35) are nonzero and equal to

$$\Sigma'_\mu \Lambda'^\mu = g, \quad \Sigma'_\mu A'^\mu = h, \quad (36)$$

$$\Sigma'^2_\mu - \Lambda'^2_\mu + e \Lambda'_\mu A'^\mu = k, \quad A'_\mu (\Lambda'^\mu - e A'^\mu) = j, \quad (37)$$

where  $g, h, j, k$  are arbitrary functions. Using gauge transformations, we express these fields by new ones

$$\Sigma' = \Sigma, \quad \Lambda' = \Lambda + ef, \quad A'_\mu = A_\mu + f_\mu. \quad (38)$$

Thus,

$$\Sigma_\mu \Lambda^\mu + ef^\mu \Sigma_\mu = g, \quad \Sigma_\mu A^\mu + f^\mu \Sigma_\mu = h, \quad (39)$$

$$\Sigma_\mu^2 - \Lambda_\mu^2 + e \Lambda_\mu A^\mu + ef^\mu (e A_\mu - \Lambda_\mu) = k, \quad (40)$$

$$A_\mu (\Lambda^\mu - e A^\mu) + f^\mu (\Lambda_\mu - e A_\mu) = j. \quad (41)$$

Additionally, the new fields are assumed to obey the strong conditions. Then we find

$$ef^\mu \Sigma_\mu = g, \quad f^\mu \Sigma_\mu = h, \quad (42)$$

$$ef^\mu (e A_\mu - \Lambda_\mu) = k, \quad f^\mu (\Lambda_\mu - e A_\mu) = j. \quad (43)$$

These formulae are not self-contradictory if

$$eh = g \wedge -ej = k. \quad (44)$$

It means that

$$\Sigma_\mu (\Lambda^\mu - e A^\mu) = 0, \quad (45)$$

$$\Sigma_\mu^2 = (\Lambda_\mu - e A_\mu)^2. \quad (46)$$

These conditions are gauge independent and can be rewritten in an elegant form like  $(D_\mu u)^2 = 0$ , as announced in equation (14), which is nothing else but the  $U(1)$  generalization of the strong integrability condition known for models with a two-dimensional target space.

Of course, the strong integrability condition (14) can be defined for all values of the coupling constant  $\lambda$ . Nonetheless, except in the case when  $\lambda = 0$ , it does not give rise to more conserved currents than in the weak sector.

The fields which obey (14) obey (13) as well. Therefore, the strong sector is a subset of the weak sector.

### 2.5. The Bogomolny sector

It is a well-known fact that for a special value of the coupling  $\lambda = 1$  (we set  $e = m = 1$ ), there is a Bogomolny sector in the Abelian Higgs model [13] in (2+1) dimensions. This sector is defined by the static, first-order differential equations

$$F_{12} + \frac{1}{2}(|u|^2 - 1) = 0, \quad (47)$$

$$(D_1 u + iD_2 u) = 0, \quad (48)$$

where we assumed  $A_0 = 0$ . Static, multi-vortex solutions of these equations as well as their dynamics have been investigated by many authors [13–25].

Let us now discuss the Bogomolny sector and its role from the integrability point of view.

First of all, the second Bogomolny equation (48) is a square root of the strong integrability condition. In fact, for static configurations we have

$$-(D_\mu u)^2 = (D_i u)^2 = (D_1 u)^2 + (D_2 u)^2 = (D_1 u + iD_2 u)(D_1 - iD_2 u). \quad (49)$$

Thus, the Bogomolny sector is a subset of the strong as well as weak integrable sector. Moreover, after the polar decomposition formula (48) reads

$$\Sigma_1 - \Lambda_2 + eA_2 = 0, \quad \Sigma_2 + \Lambda_1 - eA_1 = 0, \quad (50)$$

i.e.

$$\Lambda_i - eA_i = -\epsilon_{ij} \Sigma_j, \quad i, j = 1, 2. \quad (51)$$

It is worth stressing that the Bogomolny sector supports the existence of an infinite set of nontrivial conservation laws as conjectured by Maison [26]. As the coupling constant is non-zero, only the weak currents given by formula (15) are conserved. Then, the nonzero components

$$j_k = iG'(uu^*)uu^*(\Lambda_k - eA_k) \quad (52)$$

take the form

$$j_k = -iG'(e^{2\Sigma})e^{2\Sigma}\epsilon_{kl}\Sigma_l. \quad (53)$$

In accordance with our former discussion, the currents are identically conserved for an arbitrary function  $\Sigma$ .

An analogous analysis can be performed for the new weak currents of section 2.3. We get

$$j_k = \frac{H'}{2}F_{kl}\Sigma_l. \quad (54)$$

In this case, we use the first Bogomolny equation and obtain

$$j_k = -\frac{H'}{4}(e^{2\Sigma} - 1)\epsilon_{kl}\Sigma_l. \quad (55)$$

As we see, because of the arbitrariness of the functions  $G$  and  $H$ , both types of currents lead to identical expressions for the Bogomolny configurations. In other words, in the Bogomolny sector the two families of currents in the weak sector are undistinguishable.



### 3. Generalizations

It is straightforward to generalize our results to other models with a  $U(1)$  gauge symmetry. Some specific examples are discussed below.

#### 3.1. Generalized $U(1)$ $CP^1$ model: type I

A possible generalization of the  $U(1)$   $CP^1$  model is provided by assuming a more general form of the kinetic part for the scalar field. Namely,

$$L = -F_{\mu\nu}^2 + f(uu^*)(D_\mu u)(D^\mu u)^* - V(uu^*). \quad (56)$$

One can prove that the weak as well as the strong sectors are defined via the same formulas as in the previous section. The only difference is in the form of the strong currents which should be

$$j_\mu = \frac{i}{f(uu^*)}(G_u \pi_\mu^* - G_{u^*} \pi_\mu) \quad (57)$$

(see the remark in section 2.1). As the most prominent example of this class of Lagrangians let us mention the gauged  $O(3)$  sigma model [27], for which  $f = V = 4/(1 + |u|^2)^2$ .

#### 3.2. Generalized $U(1)$ $CP^1$ model: type II

In this case, the Lagrangian is given by the formula [28]

$$L = -g(uu^*)F_{\mu\nu}^2 + \frac{1}{2}(D_\mu u)(D^\mu u)^* - V(uu^*). \quad (58)$$

The weak sector and currents remain unchanged. However, there are no new currents in the strong sector, even if we neglect the potential  $V$ . Indeed, the first term in (58) effectively plays the role of a potential for the scalar field. The pertinent Bogomolny solutions (if they exist) form a subset of the weak sector.

#### 3.3. Abelian Chern–Simons–Higgs model

Analogously, the same integrable structure appears in the Abelian Chern–Simons–Higgs model [29–32]

$$L = \frac{\kappa}{4}\epsilon^{\alpha\beta\gamma} A_\alpha F_{\beta\gamma} + \frac{1}{2}(D_\mu u)^*(D^\mu u) - \frac{\lambda}{8}|u|^2(|u|^2 - v^2)^2. \quad (59)$$

Once again, the conditions for the Bogomolny sector imply the strong integrability conditions, and the strong conditions imply the weak ones. The Bogomolny sector exists only for the critical value of the coupling, where  $\lambda = \frac{e^4}{\kappa^2}$ .

#### 3.4. Model with a non-minimal coupling

Finally, let us comment on a model with a non-minimal coupling between the scalar and gauge fields, where the scalar field is not gauged,

$$L = -\frac{1}{4}g(uu^*)F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}u_\mu^*u^\mu - V(uu^*). \quad (60)$$

The integrable structure remains unchanged. However, there are two families of new currents which are conserved if the equation of motion for the gauge field is obeyed.

$$j_\mu^{(1)} = H^{(1)}(uu^*)gF_{\mu\nu}[u^*u^\nu + uu^{*\nu}], \quad (61)$$

$$j_{\mu}^{(2)} = H^{(2)} \left( \frac{u}{u^*} \right) g F_{\mu\nu} \left[ \frac{u^{\nu}}{u} - \frac{u^{*\nu}}{u^*} \right], \quad (62)$$

i.e.

$$j_{\mu}^{(1)} = H^{(1)}(\Sigma) g F_{\mu\nu} \Sigma^{\nu}, \quad j_{\mu}^{(2)} = H^{(2)}(\Lambda) g F_{\mu\nu} \Lambda^{\nu}, \quad (63)$$

where  $H^{(1)}$  and  $H^{(2)}$  are arbitrary real functions of the modulus and the phase respectively (the coupling function  $g$  in the first formula can be removed by a redefinition of the  $H^{(1)}$  function). Note that unlike for the models with local gauge symmetry, no additional conditions have to be imposed to provide the conservation of these currents. Therefore, there should exist symmetries in this model such that these conserved currents are their Noether currents (when additional integrability conditions have to be imposed in order to have conserved currents; this need not be true, see [33]).

#### 4. Conclusions

Let us briefly summarize the obtained results.

The property of integrability, known from nonlinear theories with global solitons, can be extended to models with an Abelian gauge group. We have established the gauge invariant description of the weak as well as strong integrable subsectors. However, in spite of the fact that both integrability conditions are gauge independent, only the weak currents do not change under gauge transformations. This rather profound difference might indicate that the weak integrable sector plays a more important role than the strong one, at least for Lagrangians with local gauge symmetry.

On the other hand, the importance of the strong integrability condition emerges from the observation that it is a Lorentz invariant generalization of the Bogomolny equations.

To conclude, the structure of the integrable sectors for all models analysed above is as follows:

$$\text{Weak sector} \supset \text{Strong sector} \supset \text{Bogomolny sector},$$

where the last inclusion makes sense only if the corresponding Bogomolny sector exists. In the cases investigated in this paper, the Bogomolny sector only exists for values of the coupling constants such that the strong sector and the Bogomolny sector do not give rise to more conserved currents than the weak sector. Still, the weak sector provides infinitely many conserved currents; therefore we may specifically conclude that in the Abelian Higgs model and related models, each Bogomolny configuration possesses infinitely many conserved currents.

In cases when there is no Bogomolny sector (e.g., for noncritical values of the couplings, or in higher dimensions), the weak and strong integrable sectors still exist and provide solutions with infinitely many conservation laws. This may indicate that in higher dimensions integrable submodels can play a role similar to the Bogomolny sectors, providing a useful tool for the investigation of such models and for the construction of solutions.

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